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On a result of Ozawa concerning uniqueness of meromorphic functions II

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Abstract

We prove a uniqueness theorem for meromorphic functions sharing three values with some finite weight which improves a recent result of the author.

Keywords: Meromorphic functions; Weighted sharing; Uniqueness

1. Introduction, definitions, and results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [1].

Definition 1. We denote by $N(r, a; f \mid = 1)$ the counting function of simple a-points of f for $a \in \mathbb{C} \cup \{\infty\}$.

Definition 2 [6]. Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid \geqslant p)$ the counting function of those a-points of f whose multiplicities are not less than p,

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where an a-point is counted according to its multiplicity. Also by $\overline{N}(r,a;f\mid\geqslant p)$ we denote the corresponding reduced counting function.

We denote by $N(r,a;f\mid\leqslant p)$ and $\overline{N}(r,a;f\mid\leqslant p)$ the following functions:

$$N(r,a;f\mid\leqslant p)=N(r,a;f)-N(r,a;f\mid\geqslant p+1)$$

and

$$\overline{N}(r,a;f\mid\leqslant p)=\overline{N}(r,a;f)-\overline{N}(r,a;f\mid\geqslant p+1).$$

Finally we define $\overline{N}(r,a;f\mid=p)$ as follows:

$$\begin{aligned}\overline{N}(r,a;f\mid=p)&=\overline{N}(r,a;f\mid\leqslant p)-\overline{N}(r,a;f\mid\leqslant p-1)\\&=\overline{N}(r,a;f\mid\geqslant p)-\overline{N}(r,a;f\mid\geqslant p+1).\end{aligned}$$

Definition 3 [6]. We put for $a\in\mathbb{C}\cup\{\infty\}$,

$$\begin{aligned}\delta_{1)}(a;f)&=1-\limsup_{r\to\infty}\frac{N(r,a;f\mid=1)}{T(r,f)},\\\delta_{2)}(a;f)&=1-\limsup_{r\to\infty}\frac{N(r,a;f\mid\leqslant 2)}{T(r,f)},\end{aligned}$$

and

$$\Theta_{2)}(a;f)=1-\limsup_{r\to\infty}\frac{\overline{N}(r,a;f\mid\leqslant 2)}{T(r,f)}.$$

It is known [9] that there exists at most a denumerable number of complex numbers $a\in\mathbb{C}\cup\{\infty\}$ for which $\delta_{1)}(a;f)>0$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_{2)}(a;f)\leqslant\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_{1)}(a;f)\leqslant 4$.

In order to investigate the influence of the distribution of zeros on the uniqueness problem of entire functions, Ozawa [7] proved the following theorem.

Theorem A [7]. *Let f,g be two entire functions of finite order. If f,g share $0,1$ CM and $2\delta(0;f)>1$ then either $f\equiv g$ or $fg\equiv 1$.*

Removing the order restriction and extending Theorem A to meromorphic functions Ueda [8] proved the following result.

Theorem B [8]. *Let f and g be two nonconstant meromorphic functions sharing $0,1,\infty$ CM. If*

$$\limsup_{r\to\infty}\frac{N(r,0;f)+N(r,\infty;f)}{T(r,f)}<\frac{1}{2},$$

then either $f\equiv g$ or $fg\equiv 1$.

Yi [10] further improved Theorem B and proved the following theorem.

Theorem C [10]. *Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM. If*

$$N(r, 0; f \mid = 1) + N(r, \infty; f \mid = 1) < \{\lambda + o(1)\} T(r, f) \quad \text{for } r \in I,$$

where $0 < \lambda < 1/2$ and I is a set of infinite linear measure, then either $f \equiv g$ or $fg \equiv 1$.

Considering $f = (e^z - 1)^2$ and $g = e^z - 1$ we see that in Theorem C the sharing of 0 cannot be relaxed from CM to IM.

In [2] the following problem is considered: *Is it possible to relax the nature of sharing 0 in Theorem* C *and if possible how far*?

To deal this problem the notion of a gradation of sharing of values is introduced in [2,3], called weighted sharing, which measures how close a shared value is to be shared IM or CM.

Definition 4 [2,3]. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity m $(\leqslant k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\leqslant k)$ and z_0 is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leqslant p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

In [2] the following result is proved.

Theorem D [2]. *Let f, g be two nonconstant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞). If*

$$N(r, 0; f \mid = 1) + N(r, \infty; f \mid = 1) < \{\lambda + o(1)\} T(r, f) \quad \text{for } r \in I,$$

where $0 < \lambda < 1/2$ and I is a set of infinite linear measure, then either $f \equiv g$ or $fg \equiv 1$.

Improving the above theorem following result is proved in [6].

Theorem E [6]. *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞). If*

$$A_0 = 2\delta_{1)}(0; f) + 2\delta_{1)}(\infty; f) + \min\left\{ \sum_{a \neq 0,1,\infty} \delta_{2)}(a; f), \sum_{a \neq 0,1,\infty} \delta_{2)}(a; g) \right\} > 3,$$

then either $f \equiv g$ or $fg \equiv 1$. If f has at least one zero or pole the case $fg \equiv 1$ does not arise.

The purpose of the paper is to investigate the possibility of reducing the weight of sharing 1 and ∞ in Theorem E to some finite weight.

Theorem 1. *Theorem* E *holds if* f, g *share* $(0,1)$, $(1,m)$, (∞,k), *where* $(m-1)(km-1) > (1+m)^2$.

We note that if f, g share $(1,\infty)$ (or (∞,∞)), it is possible to choose a sufficiently large m (or k) for which the hypothesis of the theorem is satisfied.

Following example shows that the condition $A_0 > 3$ is sharp for Theorem 1.

Example 1 [5]. Let $f = e^z - 1$ and $g = 2 - 2/e^z$. Then f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞), and $\delta_{1)}(0; f) = 0$, $\delta_{1)}(\infty; f) = 1$, $\sum_{a\neq 0,1,\infty}\delta_{2)}(a; f) = \sum_{a\neq 0,1,\infty}\delta_{2)}(a; g) = 1$. Also neither $f \equiv g$ nor $fg \equiv 1$.

Following example shows that in Theorem 1 sharing $(0,1)$ cannot be relaxed to sharing $(0,0)$.

Example 2 [2]. Let $f = (e^z - 1)^2$ and $g = e^z - 1$. Then f, g share $(0,0)$, $(1,\infty)$, (∞,∞). Also $\delta_{1)}(0; f) = \delta_{1)}(\infty; f) = 1$ but neither $f \equiv g$ nor $fg \equiv 1$.

Following example shows that Theorem 1 does not hold when $(m-1)(km-1) = (1+m)^2$.

Example 3. Let $f = 4e^z/(1+e^z)^2$, $g = 2e^z/(1+e^z)$, and $m = k = 0$. Then f, g share $(0,\infty)$, $(1,m)$, (∞,k), and $\delta_{1)}(0; f) = \delta_{1)}(\infty; f) = 1$, $(m-1)(km-1) = (1+m)^2$. Also neither $f \equiv g$ nor $fg \equiv 1$.

Following example shows that in Theorem 1, A_0 cannot be replaced by any one of the following larger quantities B_0 and C_0:

$$\begin{aligned}
B_0 &= 2\delta_{1)}(0; f) + 2\delta_{1)}(\infty; f) + \max\left\{\sum_{a\neq 0,1,\infty}\delta_{1)}(a; f), \sum_{a\neq 0,1,\infty}\delta_{1)}(a; g)\right\} \\
&\quad + \max\left\{\delta_{1)}(1; f), \delta_{1)}(1; g)\right\}, \\
C_0 &= 2\delta_{1)}(0; f) + 2\delta_{1)}(\infty; f) + \max\left\{\sum_{a\neq 0,1,\infty}\Theta_{2)}(a; f), \sum_{a\neq 0,1,\infty}\Theta_{2)}(a; g)\right\} \\
&\quad + \max\left\{\delta_{1)}(1; f), \delta_{1)}(1; g)\right\}.
\end{aligned}$$

Example 4 [4]. Let $f = e^z(1-e^z)$ a nd $g = e^{-z}(1-e^{-z})$. Then f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞). Since $f - 1/4 = -(e^z - 1/2)^2$, it follows that $B_0 > 3$ and $C_0 > 3$ but neither $f \equiv g$ nor $fg \equiv 1$. Here we note that $A_0 = 3$.

Corollary 1. *Theorem* 1 *holds for any one of the following pairs of values of* m *and* k:

(i) $m = 2$, $k = 6$, (ii) $m = 3$, $k = 4$,
(iii) $m = 4$, $k = 3$, (iv) $m = 6$, $k = 2$.

Throughout the paper we denote by f, g two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$.

2. Lemmas

In this section we present some lemmas which are required in the sequel.

Lemma 1 [6]. *If* f, g *share* $(0,0)$, $(1,0)$, $(\infty,0)$, *then*

(i) $T(r,f) \leqslant 3T(r,g) + S(r,f)$,
(ii) $T(r,g) \leqslant 3T(r,f) + S(r,g)$.

Lemma 1 shows that $S(r,f) = S(r,g)$ and we denote them by $S(r)$.

Lemma 2. *Let* f, g *share* $(0,1)$, $(1,m)$, (∞,k), *and* $f \not\equiv g$, *where* $(m-1)(km-1) > (1+m)^2$. *Then*

(i) $\overline{N}(r,0; f \mid \geqslant 2) = \overline{N}(r,1; f \mid \geqslant 2) = \overline{N}(r,\infty; f \mid \geqslant 2) = S(r)$,
(ii) $\overline{N}(r,0; g \mid \geqslant 2) = \overline{N}(r,1; g \mid \geqslant 2) = \overline{N}(r,\infty; g \mid \geqslant 2) = S(r)$.

Proof. From the given condition it is clear that $m \geqslant 2$ and $k \geqslant 2$. Also if f, g share (∞,∞) (or $(1,\infty)$) then we can choose a sufficiently large positive integer k (or m) for which the hypothesis of the lemma holds. So without loss of generality we can assume k, m to be finite. Let

$$\phi_1 = \frac{f'}{f(f-1)} - \frac{g'}{g(g-1)} = \left(\frac{f'}{f-1} - \frac{g'}{g-1}\right) - \left(\frac{f'}{f} - \frac{g'}{g}\right),$$
$$\phi_2 = \frac{f'}{f-1} - \frac{g'}{g-1} \quad \text{and} \quad \phi_3 = \frac{f'}{f} - \frac{g'}{g}.$$

We suppose that $\overline{N}(r,a;f) \neq S(r)$ for $a = 0, 1, \infty$. Since $f \not\equiv g$, it follows that $\phi_i \not\equiv 0$ for $i = 1, 2, 3$.

Now

$$\begin{aligned}\overline{N}(r,0; f \mid \geqslant 2) &= N(r,0;\phi_2) \leqslant T(r,\phi_2) + O(1) = N(r,\infty;\phi_2) + S(r)\\ &\leqslant \overline{N}(r,1; f \mid \geqslant m+1) + \overline{N}(r,\infty; f \mid \geqslant k+1) + S(r). \qquad (1)\end{aligned}$$

We note that (1) is obvious if $\overline{N}(r,0;f) = S(r)$. Again

$$
\begin{aligned}
m\overline{N}(r,1;f\mid\geqslant m+1) &\leqslant (m-1)\overline{N}(r,1;f\mid\geqslant m+1)+\overline{N}(r,1;f\mid\geqslant 2)\\
&\leqslant N(r,0;\phi_3)\leqslant N(r,\infty;\phi_3)+S(r)\\
&\leqslant \overline{N}(r,0;f\mid\geqslant 2)+\overline{N}(r,\infty;f\mid\geqslant k+1)+S(r).
\end{aligned}\tag{2}
$$

We note that (2) is obvious if $\overline{N}(r,1;f)=S(r)$. Also

$$
\begin{aligned}
k\overline{N}(r,\infty;f\mid\geqslant k+1) &\leqslant (k-1)\overline{N}(r,\infty;f\mid\geqslant k+1)+\overline{N}(r,\infty;f\mid\geqslant 2)\\
&\leqslant N(r,0;\phi_1)\leqslant N(r,\infty;\phi_1)+S(r)\\
&\leqslant \overline{N}(r,1;f\mid\geqslant m+1)+\overline{N}(r,0;f\mid\geqslant 2)+S(r).
\end{aligned}\tag{3}
$$

Clearly (3) is obvious if $\overline{N}(r,\infty;f)=S(r)$. From (1) and (2) we get

$$
\left(1-\frac{1}{m}\right)\overline{N}(r,0;f\mid\geqslant 2)\leqslant\left(1+\frac{1}{m}\right)\overline{N}(r,\infty;f\mid\geqslant k+1)+S(r).\tag{4}
$$

From (2) and (3) we obtain

$$
\left(k-\frac{1}{m}\right)\overline{N}(r,\infty;f\mid\geqslant k+1)\leqslant\left(1+\frac{1}{m}\right)\overline{N}(r,0;f\mid\geqslant 2)+S(r).\tag{5}
$$

From (4) and (5) we get

$$
\left\{\left(1-\frac{1}{m}\right)\left(k-\frac{1}{m}\right)-\left(1+\frac{1}{m}\right)^2\right\}\overline{N}(r,0;f\mid\geqslant 2)\leqslant S(r)
$$

and so

$$
\overline{N}(r,0;f\mid\geqslant 2)=S(r).
$$

Now from (5) we see that

$$
\overline{N}(r,\infty;f\mid\geqslant k+1)=S(r).
$$

Again from (2) and (3) we get

$$
\begin{aligned}
&(k-1)\overline{N}(r,\infty;f\mid\geqslant k+1)+\overline{N}(r,\infty;f\mid\geqslant 2)\\
&\quad\leqslant\left(1+\frac{1}{m}\right)\overline{N}(r,0;f\mid\geqslant 2)+\frac{1}{m}\overline{N}(r,\infty;f\mid\geqslant k+1)+S(r)=S(r)
\end{aligned}
$$

and so

$$
\overline{N}(r,\infty;f\mid\geqslant 2)=S(r).
$$

Finally from (2) we obtain

$$
\begin{aligned}
&(m-1)\overline{N}(r,1;f\mid\geqslant m+1)+\overline{N}(r,1;f\mid\geqslant 2)\\
&\quad\leqslant\overline{N}(r,0;f\mid\geqslant 2)+\overline{N}(r,\infty;f\mid\geqslant k+1)+S(r)=S(r)
\end{aligned}
$$

and so

$$
\overline{N}(r,1;f\mid\geqslant 2)=S(r).
$$

Now (ii) follows from (i) because for $a=0,1,\infty$,

$$
\overline{N}(r,a;g\mid\geqslant 2)=\overline{N}(r,a;f\mid\geqslant 2).
$$

This proves the lemma. □

Lemma 3. *For a meromorphic function* f

$$\overline{N}(r,0;f') \leqslant 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

Proof. By the first fundamental theorem and Milloux theorem [1, p. 55] we get

$$\begin{aligned}\overline{N}(r,0;f') &\leqslant \overline{N}(r,0;f) + N\left(r,0;\frac{f'}{f}\right)\\ &\leqslant \overline{N}(r,0;f) + N\left(r,\infty;\frac{f'}{f}\right) + S(r,f)\\ &= 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).\end{aligned}$$

This proves the lemma. □

Lemma 4 [4]. *If* f, g *share* $(0,1)$, $(1,\infty)$, (∞,∞), *and* $f \not\equiv g$, *then for any* a $(\neq 0,1,\infty)$,

$$\overline{N}(r,a;f \mid \geqslant 3) = \overline{N}(r,a;g \mid \geqslant 3) = S(r).$$

Lemma 5. *Let* f, g *share* $(0,1)$, $(1,m)$, (∞,k), *and* $f \not\equiv g$, *where* $(m-1)(km-1) > (1+m)^2$. *Then for any* a $(\neq 0,1,\infty)$,

$$\overline{N}(r,a;f \mid \geqslant 3) = \overline{N}(r,a;g \mid \geqslant 3) = S(r).$$

Proof. Let $\alpha = (f-1)/(g-1)$ and $h = g/f$.

If α or h is constant then clearly f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞), and so the result follows from Lemma 4. Therefore we suppose that α and h are nonconstant.

Since f, g share $(0,1)$, $(1,m)$, (∞,k), in view of Lemma 2 we get

$$\overline{N}(r,0;\alpha) \leqslant \overline{N}(r,\infty;f \mid \geqslant 2) + \overline{N}(r,1;f \mid \geqslant 2) = S(r),$$

$$\overline{N}(r,\infty;\alpha) \leqslant \overline{N}(r,\infty;f \mid \geqslant 2) + \overline{N}(r,1;f \mid \geqslant 2) = S(r),$$

$$\overline{N}(r,0;h) \leqslant \overline{N}(r,\infty;f \mid \geqslant 2) + \overline{N}(r,0;f \mid \geqslant 2) = S(r),$$

$$\overline{N}(r,\infty;h) \leqslant \overline{N}(r,\infty;f \mid \geqslant 2) + \overline{N}(r,0;f \mid \geqslant 2) = S(r).$$

Since $f = (1-\alpha)/(1-\alpha h)$, it follows that

$$f - a = \frac{(1-a) + \alpha(ah-1)}{1-\alpha h}.$$

Let z_0 be a zero of $f-a$ with multiplicity $\geqslant 3$. Then z_0 is a zero of

$$\frac{d}{dz}\big[1 - a + \alpha(ah-1)\big] = \alpha'\left[ah - 1 + \frac{a\alpha h'}{\alpha'}\right]$$

with multiplicity $\geqslant 2$. So z_0 is a zero of α' or z_0 is a zero of

$$\frac{d}{dz}\left[ah - 1 + \frac{a\alpha h'}{\alpha'}\right] = ah'\left[2 + \frac{\alpha h''}{\alpha' h'} - \frac{\alpha\alpha''}{(\alpha')^2}\right].$$

Therefore

$$\begin{aligned}\overline{N}(r,a;f\mid\geqslant 3) &\leqslant \overline{N}(r,0;\alpha')+\overline{N}(r,0;h')+T\left(2+\frac{\alpha h''}{\alpha' h'}-\frac{\alpha\alpha''}{(\alpha')^2}\right)\\ &\leqslant \overline{N}(r,0;\alpha')+\overline{N}(r,0;h')\\ &\quad+T\left(r,\frac{h''}{h'}\right)+2T\left(r,\frac{\alpha'}{\alpha}\right)+T\left(r,\frac{\alpha''}{\alpha'}\right)+O(1)\\ &\leqslant 2\overline{N}(r,0;\alpha')+2\overline{N}(r,0;h')+2\overline{N}(r,0;\alpha)\\ &\quad+3\overline{N}(r,\infty;\alpha)+\overline{N}(r,\infty;h)+S(r)\\ &=2\overline{N}(r,0;\alpha')+2\overline{N}(r,0;h')+S(r).\end{aligned}$$

So by Lemma 3 we get

$$\begin{aligned}\overline{N}(r,a;f\mid\geqslant 3) &\leqslant 4\overline{N}(r,0;\alpha)+2\overline{N}(r,\infty;\alpha)+4\overline{N}(r,0;h)+2\overline{N}(r,\infty;h)+S(r)\\ &=S(r).\end{aligned}$$

Similarly we can prove that $\overline{N}(r,a;g\mid\geqslant 3)=S(r)$. This proves the lemma. □

Lemma 6 [11]. *Let f,g share $(0,0)$, $(1,0)$, $(\infty,0)$, and*

$$H=\left(\frac{f}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

If $H\equiv 0$ then f,g share $(0,\infty)$, $(1,\infty)$, (∞,∞).

Lemma 7. *Let f,g share $(0,1)$, $(1,m)$, (∞,k), and $f\not\equiv g$, where $(m-1)(km-1)>(1+m)^2$. If $a_1,a_2,\dots,a_n$ be pairwise distinct complex numbers such that $a_i\neq 0,1,\infty$ $(i=1,2,\dots,n)$ and $H\not\equiv 0$, then*

$$\begin{aligned}N(r,\infty;H)\leqslant{}&\sum_{i=1}^{n}\overline{N}(r,a_i;f\mid=2)+\sum_{i=1}^{n}\overline{N}(r,a_i;g\mid=2)\\ &+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g')+S(r),\end{aligned}$$

where $\overline{N}_0(r,0;f')$ is the reduced counting function of the zeros of f' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f-a_i)$ and $\overline{N}_0(r,0;g')$ is analogously defined.

Proof. The possible poles of H occur at

(i) multiple zeros of f,g;
(ii) multiple zeros of $f-1$, $g-1$;
(iii) multiple poles of f,g;
(iv) multiple zeros of $f-a_i$, $g-a_i$ $(i=1,2,\dots,n)$;
(v) zeros of f',g' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f-a_i)$ and $g(g-1)\times\prod_{i=1}^{n}(g-a_i)$, respectively.

Since f,g share $(0,1)$, $(1,m)$, (∞,k), and all the poles of H are simple, the lemma follows from above and Lemmas 2 and 5. This proves the lemma. □

Lemma 8 [5]. *If f, g share $(1,1)$ and $H \not\equiv 0$, then*

$$N(r,1; f\,|=1) = N(r,1; g\,|=1) \leqslant N(r,H) + S(r).$$

3. Proof of Theorem 1

Let $f \not\equiv g$. We shall show that $fg \equiv 1$. We suppose that $H \not\equiv 0$. Let $a_1, a_2, \ldots, a_n$ be pairwise distinct complex numbers such that $a_i \neq 0, 1, \infty$ $(i = 1, 2, \ldots, n)$ and $A_n > 3 + 10\varepsilon$, where ε $(0 < \varepsilon < (A_0 - 3)/10)$ is given and

$$A_n = 2\delta_{1)}(0; f) + 2\delta_{1)}(\infty; f) + \max\left\{\sum_{i=1}^{n} \delta_{2)}(a_i; f), \sum_{i=1}^{n} \delta_{2)}(a_i; g)\right\}$$
$$+ \max\left\{\delta_{1)}(1; f), \delta_{1)}(1; g)\right\}.$$

By the second fundamental theorem we get in view of Lemma 2,

$$(n+1)T(r,f) \leqslant N(r,0; f\,|=1) + N(r,1; f\,|=1) + N(r,\infty; f\,|=1)$$
$$+ \sum_{i=1}^{n} \overline{N}(r,a_i; f) - N_0(r,0; f') + S(r) \tag{6}$$

and

$$(n+1)T(r,g) \leqslant N(r,0; g\,|=1) + N(r,1; g\,|=1) + N(r,\infty; g\,|=1)$$
$$+ \sum_{i=1}^{n} \overline{N}(r,a_i; g) - N_0(r,0; g') + S(r). \tag{7}$$

Adding (6) and (7) we get, because f, g share $(0,1)$, $(1,m)$, (∞,k),

$$(n+1)\left\{T(r,f) + T(r,g)\right\}$$
$$\leqslant 2N(r,0; f\,|=1) + 2N(r,\infty; f\,|=1) + N(r,1; f\,|=1) + N(r,1; g\,|=1)$$
$$+ \sum_{i=1}^{n} \overline{N}(r,a_i; f) + \sum_{i=1}^{n} \overline{N}(r,a_i; g) - \overline{N}_0(r,0; f') - \overline{N}_0(r,0; g') + S(r). \tag{8}$$

If $\delta_{1)}(1; g) \geqslant \delta_{1)}(1; f)$, by Lemmas 1, 5, 7, and 8 we get from (8),

$$(n+1)\left\{T(r,f) + T(r,g)\right\}$$
$$\leqslant 2N(r,0; f\,|=1) + 2N(r,\infty; f\,|=1) + \sum_{i=1}^{n} N(r,a_i; f\,|\leqslant 2)$$
$$+ \sum_{i=1}^{n} N(r,a_i; g\,|\leqslant 2) + N(r,1; g\,|=1) + S(r)$$
$$< \left\{2 - 2\delta_{1)}(0; f) + \varepsilon\right\}T(r,f) + \left\{2 - 2\delta_{1)}(\infty; f) + \varepsilon\right\}T(r,f)$$

$$+\sum_{i=1}^{n}\bigl\{1-\delta_{2)}(a_i; f)+\varepsilon/n\bigr\}T(r,f)+\sum_{i=1}^{n}\bigl\{1-\delta_{2)}(a_i; g)+\varepsilon/n\bigr\}T(r,g)$$
$$+\bigl\{1-\delta_{1)}(1; g)+\varepsilon\bigr\}T(r,g)+S(r,g),$$

i.e.,

$$\left\{2\delta_{1)}(0; f)+2\delta_{1)}(\infty; f)+\sum_{i=1}^{n}\delta_{2)}(a_i; f)-3-3\varepsilon\right\}T(r,f)$$
$$+\left\{\delta_{1)}(1; g)+\sum_{i=1}^{n}\delta_{2)}(a_i; g)-2\varepsilon\right\}T(r,g)\leqslant S(r,g),$$

i.e.,

$$\left\{2\delta_{1)}(0; f)+2\delta_{1)}(\infty; f)+\sum_{i=1}^{n}\delta_{2)}(a_i; f)-3-3\varepsilon\right\}T(r,g)$$
$$+3\left\{\sum_{i=1}^{n}\delta_{2)}(a_i; g)+\delta_{1)}(1; g)-2\varepsilon\right\}T(r,g)\leqslant S(r,g),$$

i.e.,

$$\left\{A_n+2\sum_{i=1}^{n}\delta_{2)}(a_i; g)+2\delta_{1)}(1; g)-3-9\varepsilon\right\}T(r,g)\leqslant S(r,g),$$

i.e.,

$$\left\{2\sum_{i=1}^{n}\delta_{2)}(a_i; g)+2\delta_{1)}(1; g)+\varepsilon\right\}T(r,g)\leqslant S(r,g),$$

which is a contradiction.

Again if $\delta_{1)}(1; g)<\delta_{1)}(1; f)$ then noting that $N(r,1; f\mid=1)=N(r,1; g\mid=1)$ and proceeding as above we get

$$\left\{A_n-1-7\varepsilon+2\sum_{i=1}^{n}\delta_{2)}(a_i; g)\right\}T(r,g)\leqslant S(r,g),$$

i.e.,

$$\left\{2+2\sum_{i=1}^{n}\delta_{2)}(a_i; g)+3\varepsilon\right\}T(r,g)\leqslant S(r,g),$$

which is a contradiction.

Hence $H\equiv 0$ and so by Lemma 6 f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞). Now by Theorem E we get $fg\equiv 1$. This proves the theorem. □

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